SOLUTION OF TWO CONJUGATE EQUATIONS

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UDC 536.24.02

A solution to the conjugacy problem of various types of equation has been found by reduction to an integral equation; the problem arises in the study of heat- and mass-transfer and in mechanical or electric phenomena in diverse media. Conditions are found which must be imposed on given functions for which the problem has a classical solution.

It was shown [1] that the heat-transfer equation is an equation of a hyperbolic type provided the heat spreads with finite velocity. We are, therefore, interested in heat processes which take place in media with strongly differing physical properties.

The problem considered below can be interpreted as a heat and mass-transfer problem; similar problems arise, however, when oscillations in electric lines are propagated [3, 4] or in mechanical problems [2]. It should also be noted that the present article is an extension of [5] to the case in which one of the conjugate equations holds in a finite region.

The following problem is considered: to find a continuous function $u_1(x, t)$ in a region $\bar{D}_1(0 \le x \le l, 0 \le t < T)$ which satisfies the equation

$$\frac{\partial u_1}{\partial t} = a^2 \frac{\partial^2 u_1}{\partial x^2} \tag{1}$$

in the region $D_1(0 < x < l, 0 < t < T)$, the initial condition

$$u_1|_{t=0} = f(x), \tag{2}$$

and the boundary condition

$$u_1|_{x=-l} = 0, \tag{3}$$

as well as a continuous function $u_2(x, t)$ in the region $\bar{D}_2(x \ge 0, 0 \le t < T)$ which satisfies the equation

$$\beta \frac{\partial^2 u_2}{\partial t^2} + \gamma \frac{\partial u_2}{\partial t} = \frac{\partial^2 u_2}{\partial x^2} \quad (\beta, \gamma > 0) \tag{4}$$

in the region $D_2(x > 0, 0 < t < T)$, and the initial conditions

$$u_2|_{t=0} = \varphi(x), \quad \left.\frac{\partial u_2}{\partial t}\right|_{t=0} = \psi(x), \tag{5}$$

moreover, the following conjugacy conditions must hold for the functions $u_1(x, t)$ and $u_2(x, t)$

$$u_1|_{x=-0} = \mu u_2|_{x=+0}, \tag{6}$$

$$\left.\frac{\partial u_1}{\partial x}\right|_{x=-0} = \nu \left.\frac{\partial u_2}{\partial x}\right|_{x=+0}, \tag{7}$$

where μ, ν are positive constants.

Moreover, it is also assumed that the following matching conditions are satisfied:

$$f(-l) = 0, \tag{8}$$

$$f(-0) = \mu\varphi(+0), \tag{9}$$

$$f'(-0) = \nu\varphi'(+0) \tag{10}$$

Institute of Mathematics and Mechanics, Academy of Sciences of the Kazakh SSR, Alma-Ata. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 26, No. 3, pp. 462-469, March, 1974. Original article submitted January 28, 1972.

as well as the conditions

$$\mu\psi(+0) = a^2 f''(-0), \tag{11}$$

$$\nu\psi'(+0) = a^2 f'''(-0). \tag{12}$$

To find the solutions to the problem (1)-(7) an unknown function $\omega(t)$ is introduced such that the relations given below are true:

$$\left.\frac{\partial u_1}{\partial x}\right|_{x=-0} = \omega(t), \tag{13}$$

$$\left.\frac{\partial u_2}{\partial x}\right|_{x=+0} = \frac{\omega(t)}{\nu}. \tag{14}$$

Of course, if the relations (13)-(14) hold then the conjugacy condition (7) is observed and the problem under consideration is split into the following two independent problems.

Problem A. To find a solution of Eq. (1) in the region D_1 with the initial condition (2), and the boundary conditions (3) and (13).

Problem B. To find a solution of Eq. (4) with the initial conditions (5) and the boundary condition (14).

The solution of Problem A can be obtained by using finite integral transforms with respect to x and can be written as

$$u_1(x, t) = \int_{-l}^{0} G(x, x_1, t) f(x_1)\, dx_1 + a^2 \int_{0}^{t} G(x, 0, t-\tau)\, \omega(\tau)\, d\tau,$$

where

$$G(x, x_1, t) = \frac{1}{2a\sqrt{\pi t}} \sum_{n=-\infty}^{+\infty} (-1)^n \left\{ \exp\left[-\frac{(2nl + x + x_1)^2}{4a^2 t}\right] + \exp\left[-\frac{(2nl + x - x_1)^2}{4a^2 t}\right] \right\}. \tag{15}$$

The solution of Problem B can be obtained by using operational calculus and is given by

$$u_2(x, t) = \frac{\sqrt{\beta}}{2} \frac{\partial}{\partial t} \int_{x-\frac{t}{\sqrt{\beta}}}^{x+\frac{t}{\sqrt{\beta}}} \Gamma(x, \xi, t, 0)\, \Phi(\xi)\, d\xi$$

$$+ \frac{\sqrt{\beta}}{2} \int_{x-\frac{t}{\sqrt{\beta}}}^{x+\frac{t}{\sqrt{\beta}}} \Gamma(x, \xi, t, 0) \left[\Psi(\xi) + \frac{\gamma}{\beta} \Phi(\xi)\right] d\xi - \left[\int_{0}^{t-\sqrt{\beta}x} \Gamma(x, 0, t, \tau) \frac{\omega(\tau)}{\nu}\, d\tau\right] \chi(t - \sqrt{\beta} x), \tag{16}$$

where

$$\Gamma(x, \xi, t, \tau) = \exp\left[-\frac{\gamma}{2\beta}(t-\tau)\right] I_0\left(\frac{\gamma}{2\beta}\sqrt{(t-\tau)^2 - \beta(\xi - x)^2}\right),$$

$$\Phi(\xi),\ \Psi(\xi) = \begin{cases} \varphi(-\xi),\ \psi(-\xi) & \text{for } \xi \leqslant 0, \\ \varphi(\xi),\ \psi(\xi) & \text{for } \xi > 0, \end{cases}$$

$\chi(z)$ is the Heaviside unit function.

If $t < \sqrt{\beta}x$ then the last term in the formula (16) vanishes and therefore the function $u_2(x, t)$ can be written by employing the familiar formula for the solution of a Cauchy problem with even initial functions for $x < 0$, that is, for $t < \sqrt{\beta}x$ the parabolic part exerts no effect on its hyperbolic part. To determine the unknown function $\omega(t)$ one uses the conjugacy condition (6); then from (15) and (16) one obtains the following integral equation:

$$\int_0^t \left\{ \frac{a}{\sqrt{\pi(t-\tau)}} \sum_{n=-\infty}^{+\infty} (-1)^n \exp\left[-\frac{n^2 l^2}{a^2(t-\tau)}\right] + \frac{\mu}{\nu} \exp\left[-\frac{\gamma}{2\beta}(t-\tau)\right] I_0\left(\frac{\gamma}{2\beta}(t-\tau)\right) \right\} \omega(\tau)\, d\tau = F_0(t), \tag{17}$$

where

$$F_0(t) = \mu\sqrt{\beta} \left\{ \frac{\partial}{\partial t} \int_0^{\frac{t}{\sqrt{\beta}}} \varphi(\xi) \exp\left(-\frac{\gamma}{2\beta} t\right) I_0\left(\frac{\gamma}{2\beta}\sqrt{t^2 - \beta\xi^2}\right) d\xi \right.$$

$$+\int_0^{t/\sqrt{\beta}} \exp\left(-\frac{\gamma}{2\beta}t\right) I_0\left(\frac{\gamma}{2\beta}\sqrt{t^2-\beta\xi^2}\right)\left[\psi(\xi)+\frac{\gamma}{\beta}\varphi(\xi)\right]d\xi\Bigg\} - \frac{1}{a\sqrt{\pi t}}\int_{-l}^{0}\sum_{n=-\infty}^{+\infty}(-1)^n \exp\left[-\frac{(2nl+x_1)^2}{4a^2 t}\right] f(x_1)\,dx_1. \quad (18)$$

The kernel of the integral equation (17) depends on the difference of the arguments and therefore its solution should be sought by employing the operational method; however, a direct use of this method proves inadequate in view of the difficulties of changing over to the originals. The integral equation (17) can be solved by the regularization method by which it can be reduced to a Volterra integral equation of the second kind.

The term corresponding to n = 0 is considered separately in the sum appearing in the expression for the kernel (17); one then obtains

$$\int_0^t \left[\frac{a}{\sqrt{\pi(t-\tau)}} + \frac{\mu}{\nu}\exp\left[-\frac{\gamma}{2\beta}(t-\tau)\right] I_0\left(\frac{\gamma}{2\beta}(t-\tau)\right)\right]\omega(\tau)\,d\tau$$

$$= F_0(t) - \int_0^t \frac{a}{\sqrt{\pi(t-\tau)}}\sum_{n=-\infty}^{+\infty}{}' (-1)^n \exp\left[-\frac{n^2 l^2}{a^2(t-\tau)}\right]\omega(\tau)\,d\tau, \quad (19)$$

where the symbol $\sum_{n=-\infty}^{+\infty}{}'$ indicates summation over all n from $-\infty$ to $+\infty$ except for n = 0.

The integral equation

$$\int_0^t \left[\frac{a}{\sqrt{\pi(t-\tau)}} + \frac{\mu}{\nu}\exp\left(-\frac{\gamma}{2\beta}(t-\tau)\right) I_0\left(\frac{\gamma}{2\beta}(t-\tau)\right)\right]\omega(\tau)\,d\tau = h(t), \quad (20)$$

is now considered where h(t) is assumed to be a given function. Equation (20) is called a characteristic equation for the integral equation (17). The Laplace transformation is applied to Eq. (20); one then obtains by the convolution theorem

$$\bar{\omega}(p) = \left[\frac{1}{a\sqrt{p}} - \frac{\mu}{a}\bar{G}_0(p)\right] p\bar{h}(p), \quad (21)$$

where

$$\bar{G}_0(p) = \frac{1}{\mu\sqrt{p} + \nu a\sqrt{\beta p^2 + \gamma p}}.$$

The inverse transform of $\bar{G}_0(p)$ was found in [5]. It is of the form

$$G_0(t) = \int_0^\infty W(x)\exp(-xt)\,dx,$$

where

$$W(x) = \begin{cases} \dfrac{1}{\pi\sqrt{x}\,|\mu + \nu a\sqrt{\gamma - \beta x}|}, & \text{if} \quad x \leqslant \dfrac{\gamma}{\beta}, \\ \dfrac{\mu}{\pi\sqrt{x}\,[\mu^2 + \nu^2 a^2(\beta x - \gamma)]}, & \text{if} \quad x > \dfrac{\gamma}{\beta}, \end{cases}$$

with

$$|G_0(t)| \leqslant M. \quad (22)$$

Thus if h(0) = 0 one obtains from (21)

$$\omega(t) = \int_0^t \left[\frac{1}{a\sqrt{\pi\tau}} - \frac{\mu}{a}G_0(\tau)\right] h'(t-\tau)\,d\tau. \quad (23)$$

For t = 0 the right-hand side of (19) vanishes by virtue of the matching condition (9); therefore, by substituting its derivative for h'(t) in (23) one obtains for $\omega(t)$ the following Volterra integral equation of the second kind:

$$\omega(t)+\int_0^t K(t-\tau)\,\omega(\tau)\,d\tau=H(t), \tag{24}$$

where

$$K(t-\tau)=\int_0^{t-\tau}\left[\frac{1}{a\sqrt{\pi\tau_1}}-\frac{\mu}{a}G_0(\tau_1)\right]$$

$$\times\sum_{n=-\infty}^{+\infty}{}'(-1)^n\left[\frac{n^2l^2}{a\sqrt{\pi}(t-\tau-\tau_1)^{5/2}}-\frac{a}{2\sqrt{\pi}(t-\tau-\tau_1)^{3/2}}\right]\exp\left[-\frac{n^2l^2}{a^2(t-\tau-\tau_1)}\right]d\tau_1; \tag{25}$$

$$H(t)=\int_0^t\left[\frac{1}{a\sqrt{\pi\tau}}-\frac{\mu}{a}G_0(\tau)\right]F_0'(t-\tau)\,d\tau. \tag{26}$$

The kernel $K(t-\tau)$ is a bounded function since the sum in the expression for the kernel (25) does not include a term corresponding to n = 0 (in addition, the kernel possesses continuous and bounded derivatives of any order which vanish for $t=\tau$). The solution of Eq. (24) can therefore be found by successive approximations.

Thus, if $R(t-\tau)$ is the resolvent of the kernel $K(t-\tau)$ then for w(t) one obtains

$$\omega(t)=H(t)+\int_0^t R(t-\tau)\,H(\tau)\,d\tau. \tag{27}$$

Moreover, for the formula (15) to be the solution of Eq. (1) in the region D_1 with the conditions (2), (3), and (13) satisfied it is only required that the $\omega(t)$ be continuous; it is also required at the same time that there exists its continuous and bounded derivative and that the conditions

$$\omega(0)=\nu\varphi'(+0),\quad \omega'(0)=\nu\psi'(+0), \tag{28}$$

be satisfied so that the formula (16) is the solution of (4) in the region D_2 with the conditions (5) and (14) satisfied. Therefore, the conditions for the function $\omega(t)$ to possess these properties are now given. To this end it is first noticed that in view of the property of the kernel of being infinitely many times differentiable and of all its derivatives vanishing for $\tau=t$, its resolvent also possesses the same properties. Consequently, one finds by a formal differentiation of (27),

$$\omega'(t)=H'(t)+\int_0^t R'(t-\tau)\,H(\tau)\,d\tau,$$

that is, the function $\omega'(t)$ is bounded provided that the function H'(t) is also bounded.

To prove that the function K'(t) is bounded the behavior is studied of the functions $G_0(t)$, $F_0'(t)$ and of the functions related to them in the neighborhood of the point t = 0. In [5], by using an expansion of the transform $\overline{G}_0(p)$ near the point at infinity one obtains the following representations,

$$G_0(t)=\frac{1}{\nu a\sqrt{\beta}}-\frac{2\mu\sqrt{t}}{\nu^2a^2\beta\sqrt{\pi}}+O(t)$$

or

$$G_0(t)=\frac{1}{\nu a\sqrt{\beta}}+G_1(t), \tag{29}$$

where $G_1(0)=0$,

$$G_1'(t)=-\frac{\mu}{\nu^2a^2\beta\sqrt{\pi t}}+G_2(t), \tag{30}$$

where $G_2(t)=O(1)$. A change of variable is carried out in the first two integrals (18) by $\xi=(t/\sqrt{\beta})z$, the last term being written as

$$\frac{1}{a\sqrt{\pi t}}\int_{-t}^{0}\exp\left(-\frac{x_1^2}{4a^2t}\right)f(x_1)\,dx_1+\frac{1}{a\sqrt{\pi t}}\int_{-t}^{0}\sum_{n=-\infty}^{+\infty}{}'(-1)^n\exp\left[-\frac{(2nl+x_1)^2}{4a^2t}\right]f(x_1)\,dx_1.$$

Then

$$F_0'(t) = \int_{-l}^{0} \frac{\partial}{\partial t}\left[\frac{\exp\left(-\frac{x_1^2}{4a^2t}\right)}{a\sqrt{\pi t}}\right] f(x_1)\,dx_1 + \frac{dH_0^*}{dt}, \tag{31}$$

and one notices that dH_0^*/dt is a bounded function if there exists a continuous and bounded derivative of the second order of the function $\varphi(x)$ as well as of the first order for the function $\psi(x)$, and if the function $f(x)$ is continuous and bounded. In the first term of (31) the operator $\partial/\partial t$ is replaced by $a^2\,\partial^2/\partial x_1^2$ and one now integrates twice by parts; then by taking into account the matching condition (8) and that it is required that the function f(x) possesses continuous and bounded derivatives up to the second order inclusive one obtains

$$F_0'(t) = \frac{af'(-0)}{\sqrt{\pi t}} + F_1(t), \tag{32}$$

where

$$F_1(0) = \mu\left[\frac{\varphi'(+0)}{\sqrt{\beta}} + \psi(+0)\right] - a^2 f''(-0). \tag{33}$$

Proceeding in exactly the same manner one can represent the function $F_1'(t)$ as

$$F_1'(t) = \frac{a^3 f'''(-0)}{\sqrt{\pi t}} + F_2(t), \tag{34}$$

where $F_2(t)$ is a continuous and bounded function provided that there exist continuous and bounded derivatives of the fourth order for f(x), of the third order for $\varphi(x)$, and of the second order for $\psi(x)$.

In the expression (26) $F_0'(t)$ is now replaced by (32) and $G_0(t)$ by (29); then

$$H(t) = f'(-0) + \int_0^t \frac{1}{a\sqrt{\pi\tau}} F_1(t-\tau)\,d\tau - \frac{\mu f'(-0)}{\nu a\sqrt{\beta}} \int_0^t \frac{d\tau}{\sqrt{\pi\tau}}$$

$$-\mu f'(-0)\int_0^t \frac{G_1(t-\tau)}{\sqrt{\pi\tau}}\,d\tau - \frac{\mu}{\nu a^2\sqrt{\beta}}\int_0^t F_1(\tau)\,d\tau - \frac{\mu}{a}\int_0^t G_1(t-\tau)F_1(\tau)\,d\tau. \tag{35}$$

H(t) is differentiated; then by using $G_1(0) = 0$ one obtains

$$H'(t) = \frac{1}{a\sqrt{\pi t}}\left[F_1(0) - \frac{\mu f'(-0)}{\nu\sqrt{\beta}}\right] + \int_0^t \frac{1}{a\sqrt{\pi\tau}} F_1'(t-\tau)\,d\tau$$

$$-\mu f'(-0)\int_0^t \frac{G_1'(t-\tau)}{\sqrt{\pi\tau}}\,d\tau - \frac{\mu}{\nu a^2\sqrt{\beta}} F_1(t) - \frac{\mu}{a}\int_0^t G_1'(t-\tau)F_1(\tau)\,d\tau. \tag{36}$$

It can now be seen that the function $\omega(t)$ has a bounded derivative if and only if the relation

$$F_1(0) - \frac{\mu f'(-0)}{\nu\sqrt{\beta}} = 0,$$

holds; if $F_1(0)$ is now replaced by its expression in (33) and bearing in mind the matching condition (10) one obtains the relation

$$\mu\psi(+0) - a^2 f''(-0) = 0,$$

which is identical with (11). The validity of the relations (28) is now verified. For this one proceeds in[35] to the limit with $t \to 0$; then one obviously has $H(0) = f'(-0)$ or by employing (27) and the matching condition (10) one obtains the first relation in (28). To verify the second relation of (28) one replaces $G_1'(t)$ in (36) by (30) and proceeds to the limit with $t \to 0$; one then finds for $H'(0)$

$$H'(0) = a^2 f'''(-0) + \frac{\mu^2}{\nu^2 a^2\beta} f'(-0) - \frac{\mu}{\nu a^2\sqrt{\beta}} F_1(0).$$

In the above by replacing $F_1(0)$ by its expression given in (33) and using the matching condition (10) together with the relation (27) one has $\omega'(0) = \nu\psi'(+0)$, that is, the second relation of (28) has been proved.

Thus the formulated conjugacy problem has a classical solution if the following conditions are satisfied:

1) the functions f(x), $\varphi(x)$, $\psi(x)$ possess continuous and bounded derivatives inclusively up to the fourth, third, and second order respectively;

2) the matching conditions (8)-(10) hold as well as the conditions (11) and (12).

It can be shown that (11) and (12) which connect the functions at the separation point of the equations do hold for sufficiently smooth pairs of solutions $u_1(x, t)$ and $u_2(x, t)$ in the regions $\overline{D}_1$ and $\overline{D}_2$ respectively; therefore, these conditions can, to some extent, be considered as natural.

It should be noted in conclusion that the method described here can be employed to solve a similar conjugacy problem when a second boundary condition is specified at $x = -l$.

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